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Abstract

This paper shows that in some axioms regarding the mixture of random variables, the requirement that the conclusions hold for all values of the mixture parameter can be replaced by requiring the existence of only one non-trivial value of the parameter, which needs not be fixed. This is the case for the independence, betweenness, and the mixture symmetry axioms.

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Typical mixture axioms for preferences over random variables state that "For all random variables and for all values of a mixing parameter, if some preferences hold, then other preferences hold as well." For example, the between-

such that $F'' := \alpha'' F + (1 - \alpha'')G \succ F$, a violation of , as $F' \in [F, F'']$. It follows therefore by continuity that for all $\alpha \in (0, 1)$ there is $\beta \in (0, 1)$ such that $\alpha F + (1 - \alpha)H \sim \beta G + (1 - \beta)H$.

Let $F \succ G \succ H$. By , there is an decreasing sequence α_n such that $\alpha_n F + (1 - \alpha_n)H \sim \alpha_n G + (1 - \alpha_n)H$. Let $\bar{\alpha} = \lim_{n \to \infty} \alpha_n$ (it exists as $\{\alpha_n\}$ is a decreasing and bounded sequence). By , there is $\alpha < \bar{\alpha}$ such that $\alpha F + (1 - \alpha)H \sim \alpha G + (1 - \alpha)H$. Choose therefore a sequence such that $\bar{\alpha} = 0$.

Suppose now that for a certain $\tilde{\alpha} \in (0, 1)$ there is $\tilde{\beta} \in (0, 1)$, $\tilde{\beta} \neq \tilde{\alpha}$, such that $\tilde{F} := \tilde{\alpha} F + (1 - \tilde{\alpha})H \sim \tilde{G} := \tilde{\beta} G + (1 - \tilde{\beta})H$. As before, there is a sequence $\beta_n \downarrow 0$ such that for all n, $\beta_n \tilde{F} + (1 - \beta_n)H \sim \beta_n \tilde{G} + (1 - \beta_n)H$. By construction, the line L_n through $\alpha_n F + (1 - \alpha_n)H$ and $\alpha_n G + (1 - \alpha_n)H$ and $\tilde{L}_n$ through $\beta_n \tilde{F} + (1 - \beta_n)H$ and $\beta_n \tilde{G} + (1 - \beta_n)H$ are not parallel. Wlg, H is in the interior of a probability triangle (see Machina [7]) containing also F and G. Otherwise, let $H_n \to H$ where for every n, H_n is in the interior of the triangle formed by F, G, H. The limit of the intersection points of L_n and $\tilde{L}_n$ is H, therefore these intersection points are in the triangle, a violation of transitivity, see Figure 1. ■

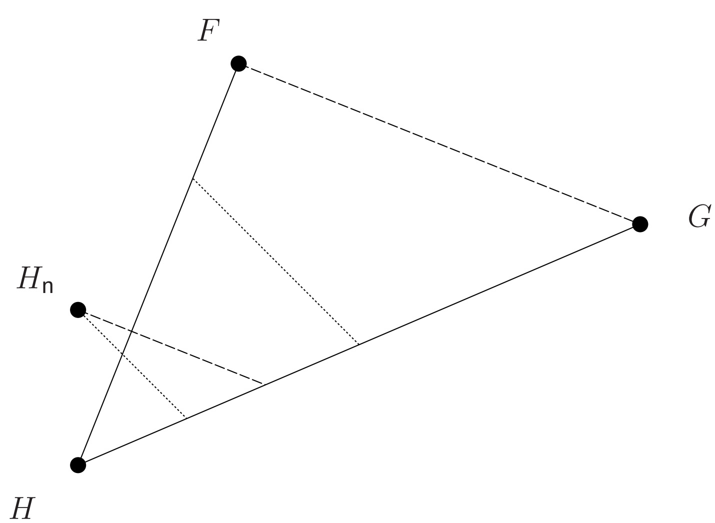
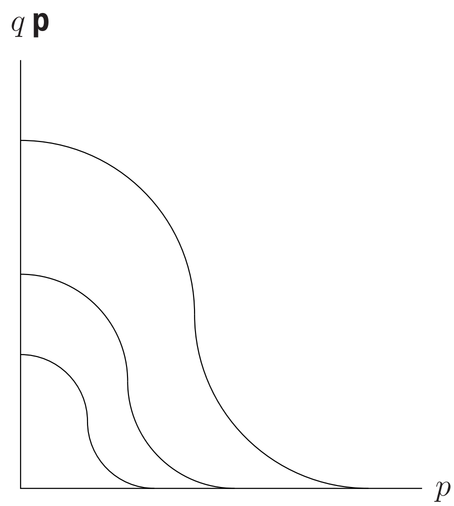


Figure 1: Wide-dash: α-lines, dense-dash: β-lines

Clearly , implies and implies on $[F, G]$ for all F and G and both and , and hence , imply .[3] However, neither , nor is implied by . For example, let □ on $□^2_+$ be represented by

$$V(p,q) = \begin{cases} \frac{2p+q+\sqrt{4pq-3q^2}}{4} & q \leqslant p \\ \frac{p^2+q^2}{2q} & q > p \end{cases}$$

(see Figure 2).



1. It satisfies
2. It satisfies
3. It can be represented by a quadratic function.

and in all three cases, it either satisfies

We obtain that $\frac{1}{2} > \alpha_1 > \overline{\alpha}$, yet $\alpha_1 F$ + (1 − α_1)G □ (1 − α_1)F + $\alpha_1 G$, in contradiction to the definition of $\overline{\alpha}$ (see eq. (1)). It thus follows that $\overline{\alpha}$ = $\frac{1}{2}$.

Next we show that for all α ⊟ $\frac{1}{2}$, $\frac{1}{2}F$ + $\frac{1}{2}G$ □ αF + (1 − α)G. Suppose not. Wlg, there is $\alpha < \frac{1}{2}$ such that αF + (1 − α)G □ $\frac{1}{2}F$ + $\frac{1}{2}G$, and since □ is on $[F, G]$, there is $\alpha < \frac{1}{2}$ such that αF + (1 − α)G □ $\frac{1}{2}F$ + $\frac{1}{2}G$. It follows that □ is decreasing in α on [β, 1] for some $\beta < \frac{1}{2}$, in contradiction to the above conclusion that $\overline{\alpha}$ = $\frac{1}{2}$. It thus follows that □ is increasing in α on [0

$\beta F + (1 - \beta)G$ □ $(1 - \gamma)F + \gamma G$ i□ $\beta = \gamma$. Let β □ $(0, 1)$ such that $\alpha F + (1 - \alpha)H = \beta F + (1 - \beta)G$, hence

$$\alpha F + (1 - \alpha)H = \beta F + (1 - \beta)[\alpha_0 F + (1 - \alpha_0)H] =$$

$$[\beta + (1 - \beta)\alpha_0]F + (1 - \beta)(1 - \alpha_0)H \ = □$$

$$\beta = \ \alpha - \alpha_0$$

by continuity, if such points exist then we can find such pairs that are not all on the same line. Therefore we can assume wlg that $[F, F'] \cap [G, G'] = \varnothing$, otherwise $[F, G'] \cap [G, F'] = \varnothing$ and the roles of G and G' are reversed. By assumption, $\frac{1}{2}F + \frac{1}{2}G \square F \square G$ while $F' \square G' \square \frac{1}{2}F' + \frac{1}{2}G'$. By continuity, for every $\alpha \in (0, 1)$ there exist $\beta_\alpha \in (0, 1)$ such that $\alpha F + (1 - \alpha)F' \square \beta_\alpha G + (1 - \beta_\alpha)G'$. By continuity, there is α such that

$$\alpha F + (1 - \alpha)F' \square \beta_\alpha G + (1 - \beta_\alpha)G' \square$$

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