

Topology Qual, Algebraic Topology:
Summer 2012

- (1) Let Σ_g denote the closed, orientable, surface of genus g .
 Prove that if Σ_h is a covering space of Σ_g , then there is a $d \in \mathbb{Z}^+$ satisfying
- $$g = d(h - 1) + 1.$$
- (2) Let X be a closed (i.e., compact & boundaryless), orientable n -dimensional manifold. Prove that if $H_{k-1}(X; \mathbb{Z})$ is torsion-free, then so is $H_k(X; \mathbb{Z})$.
- (3) Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus, concretely identified as the quotient space of the Euclidean plane by the standard integer lattice. Then any 2×2 integer matrix A induces a map

$$\phi : (\mathbb{R}/\mathbb{Z})^2 \rightarrow (\mathbb{R}/\mathbb{Z})^2$$

by left (matrix) multiplication.

- (a) Show that (with respect to a suitable basis) the induced contravariant map

$$\phi^* : H^1(T^2; \mathbb{Z}) \rightarrow H^1(T^2; \mathbb{Z})$$

on the cellular cohomology is left multiplication by the transpose of A .

- (b) Since T^2 is a closed, oriented manifold, it has a fundamental class, $[T^2] \in H_2(T^2; \mathbb{Z})$. Prove that

$$\phi_*[T^2] = \det(A) [T^2].$$

(Hint: Use part (a) and the naturality of the cup product under induced maps on homology/cohomology.)

- (4) The closed, orientable surface Σ_g of genus g , embedded in \mathbb{R}^3 in the standard way, bounds a compact region R (often called a genus g solid handlebody).
 Two copies of R , glued together by the identity map between their boundary surfaces, form a closed 3-manifold X . Compute $H_*(X; \mathbb{Z})$.

GT Qual 2012 (Spring) Part II

Show All Relevant Work!

1) Consider stereographic projection of the unit circle S^1 in \mathbf{R}^2 to \mathbf{R} from the North Pole $(0, 1)$ and from the South Pole $(0, -1)$.

a) Show that $\sigma^{-1}(x) = \frac{1-x^2}{1+x^2}$

b) Consider the smooth vector field $\frac{d}{dx}$ on \mathbf{R} . Using σ , this induces a smooth vector field on the circle minus the North Pole. Can it be extended to a smooth vector field on all of S^1 ?

2a) A smooth map $F : M \rightarrow N$ is a *submersion* if...

b) Let M be a compact, smooth 3-manifold. Prove that there is no submersion $F : M \rightarrow \mathbf{R}^3$.

3) Consider D the open unit disk in \mathbf{R}^2 with Riemannian metric

$$g = \left(\frac{2}{1+x^2+y^2}\right)^2 dx^2 + \left(\frac{2}{1+x^2+y^2}\right)^2 dy^2$$

a) Write down an (oriented) orthonormal frame $(E_1; E_2)$ for D with respect to this metric.

b) Write down the associated dual coframe $(\theta^1; \theta^2)$.

c) Compute $\int_D \theta^1 \wedge \theta^2$. Is this the Riemannian volume form (that is, does it agree with the volume formula $\sqrt{\det(g_{ij})} dx \wedge dy$)?

d) Compute the volume (area?) of D with respect to this metric.

e) What have you computed?

4) Suppose that f_0 and f_1 are smoothly homotopic maps from X to Y and that X is a