Algebra Qualifying Exam
Spring 2015
3 hours

1. (a) Show that $\mathrm{GL}_2(\mathbb{F}_5)$ has a unique conjugacy class of elements of order three.

(b) Classify, up to isomorphism, all groups of order $3 \cdot 5^2$, and give a presentation for each group. Hint: $\mathrm{Aut}(\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}5$ ).

2. Suppose $F$ is a field and $a$ $F$. For each of the following groups $G$, either find an example of $F$ and $a$ for which $x^4 - a$ $F[x]$ has Galois group $G$, or show that no such $F$ and $a$ exist.

$$G = D_8, \qquad G = S_4, \qquad G = \mathbb{Z}/4\mathbb{Z}.$$

3. Suppose $p$ is a prime. Show that the Galois group of $x^5 - 1$ $\mathbb{F}_p[x]$ depends only on $p$ (mod 5), and compute it for each congruence class $p$ (mod 5).

4. Suppose $R$ is a Noetherian local ring with maximal ideal $\mathfrak{m}$. If $\mathfrak{a}$ is an ideal such that the *only* prime ideal containing $\mathfrak{a}$ is $\mathfrak{m}$, show that $\mathfrak{m}^k$ $\mathfrak{a}$ for $k$ 0.

5. Suppose $R$ is a UFD, and let $R_\mathfrak{p}$ be the localization of $R$ at a prime $\mathfrak{p} = (\pi)$ generated



$, \ldots, x_n]/I$ is finite dimensional.
Hint: set $J = \sqrt{I}$ and prove that each $J^k/J^{k+1}$ a finite dimensional $\mathbb{C}$-vector space.

8. Let $k$ be an algebraically closed field. Let $V$ be the algebraic subset of $\mathbb{A}^2$ over $k$ cut out by the equation $y^2 = x^3 + x^2$.

(a) Show that the normalization of $k[V]$ is the polynomial ring $k[t]$ where $t = y/x$.

(b) Compute the fibers of the map $\phi : \mathbb{A}^1 \to V$ that corresponds to the inclusion $\phi^{*}$ : $k[V] \to k[t]$.